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LETTER TO THE EDITOR

A new result for Laguerre polynomials

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Abstract. We prove the following general result for Laguerre polynomials.

For all $x, \alpha \in \mathbb{C}$

$$\sum_{k=j}^{i} k^s L_{i-k}^{(-\alpha-i-1)}(-x) L_{k-j}^{(\alpha+j)}(x) = \delta_{i,2s+j}(-x)^s \qquad i,\, j,\, s \in \{0, 1, 2, \ldots\}$$

provided that $i \geqslant 2s + j$.

1. Known results

The Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ are defined by

$$L_n^{(\alpha)}(x) = \frac{1}{n!} \sum_{k=0}^{n} (-n)_k (\alpha + k + 1)_{n-k} \frac{x^k}{k!} \qquad n \in \{0, 1, 2, \ldots\}$$

for all complex α and x. For $n \in \{-1, -2, -3, \ldots\}$ we define $L_n^{(\alpha)}(x) = 0$. They are polynomials in x and in α. For α real and $\alpha > -1$ they are orthogonal on the interval $[0, \infty)$ with respect to the weight function $x^{\alpha} \mathrm{e}^{-x}$. We mention some well known formulae for these polynomials (see [1]). If $\mathrm{D} = \mathrm{d}/\mathrm{d}x$ denotes the differential operator then

$$\mathrm{D}^k L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x) \qquad k \leqslant n,\ k, n \in \{0, 1, 2, \ldots\} \tag{1}$$

and

$$[-x\mathrm{D}^2 - (\alpha + 1 - x)\mathrm{D}] L_n^{(\alpha)}(x) = n L_n^{(\alpha)}(x) \qquad n \in \{0, 1, 2, \ldots\}. \tag{2}$$

From the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1 - t)^{-\alpha-1} \exp\left(\frac{xt}{t-1}\right)$$

it is easy to obtain (see [2, 3])

$$\sum_{k=0}^{\infty} L_k^{(-\alpha-i-1)}(-x) t^k \sum_{m=0}^{\infty} L_m^{(\alpha+j)}(x) t^m = (1 - t)^{i-j-1}$$

which leads to

$$\sum_{k=j}^{i} L_{i-k}^{(-\alpha-i-1)}(-x) L_{k-j}^{(\alpha+j)}(x) = \delta_{i,j} \qquad j \leqslant i,\ i, j \in \{0, 1, 2, \ldots\}. \tag{3}$$

This gives the general result indicated in the abstract in the case $s = 0$.

2. Derivation of the formula

Let $J^{(\alpha)}(x;\, j,k)$ be the linear differential operator of the form

$$J^{(\alpha)}(x;\, j,k) = \sum_{i=1}^{\infty} j_i^{(\alpha)}(x;\, j,k)\mathrm{D}^i$$

such that for a certain value $j \in \{0,1,2,\ldots\}$, $k \in \{1,2,3,\ldots\}$, $j \leqslant k$

$$\sum_{i=1}^{\infty} j_i^{(\alpha)}(x;\, j,k)\mathrm{D}^i L_n^{(\alpha)}(x) = \delta_{n,k} L_{n-j}^{(\alpha+j)}(x) \qquad \text{for all } n \in \{1,2,3,\ldots\}.$$

The coefficients $j_i^{(\alpha)}(x;\, j,k)$ are uniquely determined and can be calculated by (3) (see [3, lemma 5])

$$\begin{aligned} j_i^{(\alpha)}(x;\, j,k) &= (-1)^i \sum_{n=j}^{i} L_{i-n}^{(-\alpha-i-1)}(-x)\delta_{n,k} L_{n-j}^{(\alpha+j)}(x) \\ &= (-1)^i L_{i-k}^{(-\alpha-i-1)}(-x) L_{k-j}^{(\alpha+j)}(x). \end{aligned}$$

For $s \in \{1,2,3,\ldots\}$ the operator

$$H^{(\alpha)}(x;\, j,s) = (-1)^j \sum_{k=\max\{i,j\}}^{\infty} k^s J^{(\alpha)}(x;\, j,k)$$

has the property that for all $n \in \{0,1,2,\ldots\}$

$$H^{(\alpha)}(x;\, j,s) L_n^{(\alpha)}(x) = (-1)^j n^s L_{n-j}^{(\alpha+j)}(x).$$

However, by (1) and (2), it is easy to see that for all $n, j \in \{0,1,2,\ldots\}$ and $s \in \{1,2,3,\ldots\}$

$$\mathrm{D}^j[-x\mathrm{D}^2 - (\alpha+1-x)\mathrm{D}]^s L_n^{(\alpha)}(x) = (-1)^j n^s L_{n-j}^{(\alpha+j)}(x).$$

It follows that

$$\mathrm{D}^j[-x\mathrm{D}^2 - (\alpha+1-x)\mathrm{D}]^s = \sum_{i=1}^{\infty}(-1)^{i+j}\left[\sum_{k=\max\{i,j\}}^{\infty} k^s L_{i-k}^{(-\alpha-i-1)}(-x) L_{k-j}^{(\alpha+j)}(x)\right]\mathrm{D}^i \qquad (4)$$

which implies the desired result for $s \in \{1,2,3,\ldots\}$.

3. Application

In some recent papers on differential operators for generalizations of Laguerre polynomials [2, 3] the coefficients of the differential operators contain expressions of the form

$$\sum_{k=j}^{i} \binom{k+\alpha+1}{k} L_{i-k}^{(-\alpha-i-1)}(-x) L_{k-j}^{(\alpha+j)}(x).$$

After tedious computations they turn out to vanish for sufficiently large values of i in the case that α is an integer greater than -1, proving that in that case the operators are of finite order. Now this is a direct consequence of the general result obtained.

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